

Adding Isolated Vertices Makes some Greedy Online Algorithms Optimal[☆]

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Abstract

An unexpected difference between online and offline algorithms is observed. The natural greedy algorithms are shown to be worst case online optimal for ONLINE INDEPENDENT SET and ONLINE VERTEX COVER on graphs with “enough” isolated vertices, Freckle Graphs. For ONLINE DOMINATING SET, the greedy algorithm is shown to be worst case online optimal on graphs with at least one isolated vertex. These algorithms are not online optimal in general. The online optimality results for these greedy algorithms imply optimality according to various worst case performance measures, such as the competitive ratio. It is also shown that, despite this worst case optimality, there are Freckle graphs where the greedy independent set algorithm is objectively less good than another algorithm.

It is shown that it is NP-hard to determine any of the following for a given graph: the online independence number, the online vertex cover number, and the online domination number.

Keywords: online algorithms, greedy algorithm, isolated vertices, online independence number

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1. Introduction

This paper contributes to the larger goal of better understanding the nature of online optimality, greedy algorithms, and different performance measures for online algorithms. The graph problems ONLINE INDEPENDENT SET, ONLINE VERTEX COVER and ONLINE DOMINATING SET, which are defined below, are considered in the *vertex-arrival* model, where the vertices of a graph, G , are revealed one by one. When a vertex is revealed (we also say that it is “requested”), its edges to previously revealed vertices are revealed. At this point, an algorithm irrevocably either accepts the vertex or rejects it. This model is well-studied (see for example, [18, 12, 21, 10, 17, 13, 14]).

We show that, for some graphs, an obvious greedy algorithm for each of these problems performs less well than another online algorithm and thus is not online optimal. However, this greedy algorithm performs (at least in some sense) at least as well as any other online algorithm for these problems, as long as the graph has enough isolated vertices. Thus, in contrast to the case with offline algorithms, adding isolated vertices to a graph can improve an algorithm’s performance, even making it “optimal”.

For an online algorithm for these problems and a particular sequence of requests, let S denote the set of accepted vertices, which we call a *solution*. When all vertices have been revealed (requested and either accepted or rejected by the algorithm), S must fulfill certain conditions:

- In the ONLINE INDEPENDENT SET problem [14, 7], S must form an independent set. That is, no two vertices in S may have an edge between them. The goal is to maximize $|S|$.
- In the ONLINE VERTEX COVER problem [8], S must form a vertex cover. That is, each edge in G must have at least one endpoint in S . The goal is to minimize $|S|$.
- In the ONLINE DOMINATING SET problem [20], S must form a dominating set. That is, each vertex in G must be in S or have a neighbor in S . The goal is to minimize $|S|$.

If a solution does not live up to the specified requirement, it is said to be infeasible. The score of a feasible solution is $|S|$. The score of an infeasible solution is ∞ for minimization problems and $-\infty$ for maximization problems. Note that for ONLINE DOMINATING SET, it is not required that S form a

dominating set at all times. It just needs to be a dominating set when the whole graph has been revealed. If, for example, it is known that the graph is connected, the algorithm might reject the first vertex since it is known that it will be possible to dominate this vertex later.

In Section 2, we define the greedy algorithms for the above problems, along with concepts analogous to the online chromatic number of Gyárfás et al. [11] for the above problems, giving a natural definition of optimality for online algorithms. In Section 3, we show that greedy algorithms are not in general online optimal for these problems. In Section 4, we define Freckle Graphs, which are graphs which have “enough” isolated vertices to make the greedy algorithms online optimal. In proving that the greedy algorithms are optimal on Freckle Graphs, we also show that, for **ONLINE INDEPENDENT SET**, one can, without loss of generality, only consider adversaries which never request a vertex adjacent to an already accepted vertex, while there are alternatives. In Section 5, we investigate what other online problems have the property that adding isolated requests make greedy algorithms optimal. In Section 6, it is shown that the online optimality results for these greedy algorithms imply optimality according to various worst case performance measures, such as the competitive ratio. In Section 7, it is shown that, despite this worst case optimality, there is a family of Freckle graphs where the greedy independent set algorithm is objectively less good than another algorithm. Various NP-hardness results concerning optimality are proven in Section 8. There are some concluding remarks and open questions in the last section. Note that Theorem 8.1 and Theorem 8.4 appeared in the second author’s Master’s thesis [15], which served as inspiration for this paper.

2. Algorithms and Preliminaries

For each of the three problems, we define a greedy algorithm.

- In **ONLINE INDEPENDENT SET**, **GIS** accepts a revealed vertex, v , iff no neighbors of v have been accepted.
- In **ONLINE VERTEX COVER**, **GVC** accepts a revealed vertex, v , iff a neighbor of v has previously been revealed but not accepted.
- In **ONLINE DOMINATING SET**, **GDS** accepts a revealed vertex, v , iff no neighbors of v have been accepted.

Note that the algorithms **GIS** and **GDS** are the same (they have different names to emphasize that they solve different problems). For an algorithm **ALG**, we define $\overline{\mathbf{ALG}}$ to be the algorithm that simulates **ALG** and accepts exactly those vertices that **ALG** rejects. This defines a bijection between **ONLINE INDEPENDENT SET** and **ONLINE VERTEX COVER** algorithms. Note that $\mathbf{GVC} = \overline{\mathbf{GIS}}$.

For a graph, G , an ordering of the vertices, ϕ , and an algorithm, **ALG**, we let $\mathbf{ALG}(\phi(G))$ denote the score of **ALG** on G when the vertices are requested in the order ϕ . We let $|G|$ denote the number of vertices in G .

For minimization problems, we define:

$$\mathbf{ALG}(G) = \max_{\phi} \mathbf{ALG}(\phi(G))$$

That is, $\mathbf{ALG}(G)$ is the highest score **ALG** can achieve over all orderings of the vertices in G .

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Since we consider a worst possible ordering, we sometimes think of an adversary as ordering the vertices.

Observation 2.1. *Let **ALG** be an algorithm for **ONLINE INDEPENDENT SET**. Let a graph, G , with n vertices be given. Now, $\overline{\mathbf{ALG}}$ is an **ONLINE VERTEX COVER** algorithm and $\mathbf{ALG}(G) + \overline{\mathbf{ALG}}(G) = n$.*

The equality $\mathbf{ALG}(G) + \overline{\mathbf{ALG}}(G) = n$ holds, since a worst ordering of G for **ALG** is also a worst ordering for $\overline{\mathbf{ALG}}$.

In considering online algorithms for coloring, [11] defines the online chromatic number, which intuitively is the best result (minimum number of colors) any online algorithm can be guaranteed to obtain for a particular graph (even when the graph, but not the ordering, is known in advance). We define analogous concepts for the problems we consider, defining for every graph a number representing the best value any online algorithm can achieve. Note that in considering all algorithms, we include those which know the graph in advance. Of course, when the graph is known, the order in which the vertices are requested is not known to an online algorithm, and the label given with

a requested vertex does not necessarily correspond to its label in the known graph: The subgraph revealed up to this point might be isomorphic to more than one subgraph of the known graph and it could correspond to any of these subgraphs.

Let $I^O(G)$ denote the *online independence number* of G . This is the largest number such that there exists an algorithm, **ALG**, for **ONLINE INDEPENDENT SET** with $\text{ALG}(G) = I^O(G)$. Similarly, let $V^O(G)$, the *online vertex cover number*, be the smallest number such that there exists an algorithm, **ALG**, for **ONLINE VERTEX COVER** with $\text{ALG}(G) = V^O(G)$. Also let $D^O(G)$, the *online domination number*, be the smallest number such that there exists an algorithm, **ALG**, for **ONLINE DOMINATING SET** with $\text{ALG}(G) = D^O(G)$.

The same relation between the online independence number and the online vertex cover number holds as between the independence number and the vertex cover number.

Observation 2.2. *For a graph, G with n vertices, we have $I^O(G) + V^O(G) = n$.*

Proof. Let a graph, G , with n vertices be given. Let **ALG** be an algorithm for **ONLINE INDEPENDENT SET** such that $\text{ALG}(G) = I^O(G)$. From Observation 2.1, we have that $\overline{\text{ALG}}$ is an algorithm for **ONLINE VERTEX COVER** such that $\overline{\text{ALG}}(G) = n - I^O(G)$. It must hold that $\overline{\text{ALG}}(G) = V^O(G)$, since the existence of an algorithm with a lower vertex cover number would imply the existence of a corresponding algorithm for **ONLINE INDEPENDENT SET** with an independence number greater than $\text{ALG}(G) = I^O(G)$. \square \square

3. Non-optimality of Greedy Algorithms

We start by motivating the other results in this paper by showing that the greedy algorithms are not optimal in general. In particular, they are not optimal on the star graphs, S_n , $n \geq 3$, which have a center vertex, s , and n other vertices, adjacent to s , but not to each other.

The algorithm, **IS-STAR** (see Algorithm 1), does much better than **GIS** for the independent set problem on star graphs.

Theorem 3.1. *For a star graph, S_n , $\text{IS-STAR}(S_n) = n - 1$ and $\text{GIS}(S_n) = 1$.*

Proof. We first show that **IS-STAR** never accepts the center vertex, s . If s is presented first, it will be rejected. If it is presented second, it will have an

Algorithm 1 IS-STAR, an online optimal algorithm for independent set for S_n

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1: for request to vertex  $v$  do
2:   if  $v$  is the first vertex then
3:     reject  $v$ 
4:   else if  $v$  is the second vertex and it has an edge to the first then
5:     reject  $v$ 
6:   else if  $v$  has more than one neighbor already then
7:     reject  $v$ 
8:   else
9:     accept  $v$ 

```

edge to the first vertex and be rejected. If it is presented later, it will have more than one neighbor and be rejected. Since IS-STAR never accepts s , it produces an independent set. For every ordering of the vertices, IS-STAR will reject the first vertex. If the first vertex is s , it will reject the second vertex. Otherwise, it will reject s when it comes. Thus, $\text{IS-STAR}(S_n) = n - 1$. On the other hand, $\text{GIS}(G) = 1$, since it will accept s if it is requested first. $\square \square$

Since $n - 1 > 1$ for $n \geq 3$, we can conclude that GIS is not an optimal online algorithm for all graph classes.

Corollary 3.2. *For ONLINE INDEPENDENT SET, there exists an infinite family of graphs, S_n for $n \geq 3$, and an online algorithm, IS-STAR, such that $\text{GIS}(S_n) < \text{IS-STAR}(S_n)$.*

Note that if some algorithm, ALG, rejects the first vertex requested, $\text{ALG}(S_n) \leq n - 1$, and if it accepts the first vertex, $\text{ALG}(S_n) = 1$. Thus IS-STAR is optimal.

To show that GVC is not an optimal algorithm for ONLINE VERTEX COVER, we consider $\overline{\text{IS-STAR}}$.

Corollary 3.3. *For ONLINE VERTEX COVER, there exists an infinite family of graphs, S_n for $n \geq 3$, and an online algorithm, $\overline{\text{IS-STAR}}$, such that $\overline{\text{IS-STAR}}(S_n) < \text{GVC}(S_n)$.*

Proof. Using Observation 2.1 and Theorem 3.1, we have that $\overline{\text{IS-STAR}}(S_n) = n + 1 - \text{IS-STAR}(S_n) = 2$ and $\text{GVC}(S_n) = n + 1 - \text{GIS}(S_n) = n$. $\square \square$

Finally, for ONLINE DOMINATING SET, we have a similar result.

Corollary 3.4. *For ONLINE DOMINATING SET, there exists an infinite family of graphs, S_n for $n \geq 3$, and an online algorithm, $\overline{\text{IS-STAR}}$, such that $\overline{\text{IS-STAR}}(S_n) < \text{GDS}(S_n)$.*

Proof. Requesting s last ensures that GDS accepts n vertices. It can never accept all $n + 1$ vertices, so $\text{GDS}(S_n) = n$. On the other hand, $\overline{\text{IS-STAR}}(S_n) = 2$ (as in the proof of Corollary 3.3). We note that a vertex cover is also a dominating set in connected graphs. This means that $\overline{\text{IS-STAR}}$ always produces a dominating set in S_n . \square \square

4. Optimality of Greedy Algorithms on Freckle Graphs

For a graph, G , we let

- k denote the number of isolated vertices,
- G' denote the graph induced by the non-isolated vertices,
- $b(G')$ be a maximum independent set in G' , and
- $s(G')$ be a minimum inclusion-maximal independent set in G' (that is, a smallest independent set such that including any additional vertex in the set would cause it to no longer be independent).

Note that $|s(G')|$ is also known as the *independent domination number* of G' (see [1] for more information).

Using this notation, we define the following class of graphs.

Definition 4.1. *A graph, G , is a Freckle Graph if $k + |s(G')| \geq I^O(G')$.*

Note that all graphs where at least half the vertices are isolated are Freckle Graphs. If the definition was changed to this (which might be less artificial), the results presented here would still hold, but our definition gives stronger results. The name comes from the idea that such a graph in many cases has a lot of isolated vertices (freckles). Furthermore, any graph can be turned into a Freckle Graph by adding enough isolated vertices. Note that a complete graph is a Freckle Graph. To make the star graph, S_n , a freckle graph, we need to add $n - 2$ isolated vertices. We show that GIS and GVC are online optimal on all Freckle Graphs. For the proof, we need a little more terminology and a helpful lemma.

Definition 4.2. A request is *pointless* if it is to a vertex which has a neighbor which was already accepted.

Definition 4.3. For a graph, G , an adversary is said to be *conservative* if it does not make pointless requests unless only such requests remain.

Lemma 4.4. For ONLINE INDEPENDENT SET, for every graph, G , there exists a conservative adversary, ADV , which ensures that every algorithm accepts an independent set in G of size at most $I^O(G)$.

Proof. Assume, for the sake of contradiction, that there exists an algorithm ALG , which accepts an independent set of size at least $I^O(G) + 1$ against every conservative adversary. We now describe an algorithm, ALG' , which accepts an independent set of size at least $I^O(G) + 1$ against any adversary. This contradicts the definition of $I^O(G)$.

Intuitively, since pointless requests must be rejected by any algorithm, ALG' can reject pointless requests and otherwise ignore them, reacting as ALG would against a conservative adversary on the other requests. ALG' works as follows: It maintains a virtual graph, G' , which, inductively, is a copy of the part of G revealed so far, but without the pointless requests. When a new non-pointless vertex is requested, the same vertex is added to G' , including only the edges to previous vertices which are not pointless (the pointless requests are not in G'). ALG' now accepts this request if ALG accepts the corresponding request in G' . When a pointless request is made, ALG' rejects it and does not add it to G' .

Note that every time ALG accepts a vertex in G' , ALG' accepts the corresponding vertex in G . Thus, $ALG'(G) \geq ALG(G') \geq I^O(G) + 1$ which is a contradiction. \square \square

Theorem 4.5. For any algorithm, ALG , for ONLINE INDEPENDENT SET, and for any Freckle Graph, G , $GIS(G) \geq ALG(G)$.

Proof. First, we note that GIS will accept the k isolated vertices. In G' , it will accept an inclusion-maximal independent set. Since we take the worst ordering, it accepts $|s(G')|$ vertices. We get $GIS(G) = k + |s(G')|$. Now we describe an adversary strategy which ensures that an arbitrary algorithm, ALG , accepts at most $k + |s(G')|$ vertices.

The adversary starts by presenting isolated vertices until ALG either accepts $|s(G')|$ vertices or rejects k vertices.

If **ALG** accepts $|s(G')|$ vertices, the adversary decides that they are exactly those in $s(G')$. This means that **ALG** will accept no other vertices in G' . Thus, it accepts at most $k + s(G')$ vertices.

If **ALG** rejects k vertices, the adversary decides that they are the k isolated vertices. We now consider G' . At this point, up to $|s(G')| - 1$ isolated vertices may have been requested and accepted. Using Lemma 4.4, we see that requesting independent vertices up to this point is optimal play from an adversary playing against an algorithm which has accepted all of these isolated requests. Following this optimal conservative adversary strategy ensures that the algorithm accepts an independent set of size at most $I^O(G') \leq k + |s(G')| = \mathbf{GIS}(G)$. \square \square

Corollary 4.6. *For any Freckle Graph, G , $\mathbf{GIS}(G) = I^O(G)$.*

Intuitively, **GIS** becomes optimal on Freckle Graphs because the isolated vertices allow it to accept a larger independent set, even though it still does poorly on the connected part of the graph. Any algorithm, which outperforms **GIS** on the connected part of the graph, must reject a large number of the isolated vertices in order to keep this advantage.

In contrast, for vertex cover adding isolated vertices to a graph does not make **GVC** accept fewer vertices. **GVC** becomes optimal on Freckle Graphs because the isolated vertices force any other online algorithm to accept some of those isolated vertices.

Corollary 4.7. *For any algorithm, \mathbf{ALG} , for **ONLINE VERTEX COVER**, and for any Freckle Graph, G , $\mathbf{GVC}(G) \leq \mathbf{ALG}(G)$.*

Proof. This follows from Theorem 4.5, Observation 2.1, and the fact that $\mathbf{GVC} = \overline{\mathbf{GIS}}$. \square \square

Corollary 4.8. *For any Freckle Graph, G , $\mathbf{GVC}(G) = V^O(G)$.*

For **ONLINE DOMINATING SET** something similar holds, but only one isolated vertex is needed. **GDS** becomes optimal because any dominating set has to include that isolated vertex.

Theorem 4.9. *For any algorithm, \mathbf{ALG} , for **ONLINE DOMINATING SET** and for any graph, G , with at least one isolated vertex, $\mathbf{GDS}(G) \leq \mathbf{ALG}(G)$.*

Proof. Recall that k denotes the number of isolated vertices in G , and G' denotes the subgraph of G induced by the non-isolated vertices. Note that

GDS always produces an independent set. Thus, GDS accepts at most $k + |b(G')|$ vertices; it accepts exactly the k isolated vertices and the vertices in $b(G')$ if these are presented first.

Let an algorithm, **ALG**, be given. The adversary can start by presenting $k + |b(G')|$ isolated vertices. If at least one of these vertices is not accepted by **ALG**, the adversary can decide that this was in fact an isolated vertex, which can now no longer be dominated. Thus, $\text{ALG}(G) = \infty$. If **ALG** accepts all the presented vertices, it gets a score of at least $k + |b(G')|$. $\square \quad \square$

Corollary 4.10. *For any graph, G , with an isolated vertex, $\text{GDS}(G) = D^O(G)$.*

5. Adding Isolated Elements in Other Problems

These results, showing that adding isolated vertices to a graph can make the greedy algorithms for **ONLINE INDEPENDENT SET** and **ONLINE VERTEX COVER** optimal, lead one to ask if similar results hold for other problems. The answer is clearly “yes”: We give similar results for **ONLINE MATCHING** and **MAXIMUM ONLINE SET** (including **ONLINE MATROID INTERSECTION** as a special case).

We consider **ONLINE MATCHING** in the edge-arrival model, so each request is an edge which must be accepted or rejected. If one or both of the vertices that are endpoints of the edge have not been revealed yet, they are revealed with the edge. The goal is to accept as large a set, S , as possible, under the restriction that S is a matching. Thus, no two edges in S can be incident to each other. One can define $M^O(G)$, the *online matching number* of G , analogously to the online independence number, to be the largest number such that there exists an algorithm, **ALG**, for **ONLINE MATCHING** with $\text{ALG}(G) = M^O(G)$. Let **GM** be the natural greedy algorithm for **ONLINE MATCHING**, which accepts any edge not incident to any edge already accepted. Instead of adding isolated vertices, we add isolated edges, edges which do not share any vertices with any other edges. The number of isolated edges to add would be k , where $M^O(G) \leq \text{GM}(G) + k$. We get the following theorem: Let G' denote the graph G induced by the non-isolated edges.

Theorem 5.1. *Let G be a graph where $M^O(G') \leq \text{GM}(G') + k$. For **ONLINE MATCHING**, we have that*

$$\text{GM}(G) = M^O(G).$$

Proof. Note that a matching in a graph $G = (V, E)$ corresponds to an independent set in the line graph $L(G)$, where the vertices of $L(G)$ correspond to the edges of G , and two vertices of $L(G)$ are adjacent, if and only if the corresponding edges are incident to each other in G . Thus, since **GIS** is optimal for the graph with $I^O(L(G)) - \mathbf{GIS}(L(G))$ isolated vertices (or more), **GM** is optimal for the graph with $M^O(G') - \mathbf{GM}(G')$ isolated edges (or more). \square \square

All of the above problems are in the class AOC [5], so one is tempted to ask if all problems in AOC have a similar property, or if all maximization problems in AOC do. This is not the case.

Definition 5.2. *A problem is in AOC (Asymmetric Online Covering) if the following hold:*

- *Each request must be either accepted or rejected on arrival.*
- *The cost (profit) of a feasible solution is the number of accepted requests.*
- *The cost (profit) of an infeasible solution is ∞ ($-\infty$).*
- *For any request sequence, there exists at least one feasible solution.*
- *A superset (subset) of a minimum cost (maximum profit) solution is feasible.*

An upper bound on the advice complexity of all problems in AOC was proven in [5], along with a matching lower bound for a subset of these problems, the AOC-complete problems.

Theorem 5.3. *There exists a maximization problem in the class AOC, where adding isolated requests which are independent of all others in the sense that these requests can be added to any feasible set, maintaining feasibility, does not make the natural greedy algorithm optimal.*

Proof. Consider the problem **ONLINE MAXIMAL FOREST**, in the vertex arrival model, where the goal is to accept as large a set, S , of vertices, as possible, under the restriction that S may not contain a cycle. Consider the following graph, $G'_n = (V, E)$, where

$$V = \{x, y, v_1, v_2, \dots, v_n\} \text{ and} \\ E = \{(x, y)\} \cup \{(x, v_i), (y, v_i) \mid 1 \leq i \leq n\}.$$

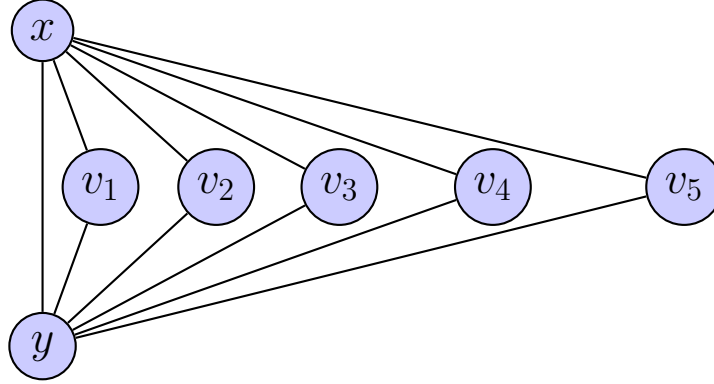


Figure 1: The graph G'_5 .

Figure 1 shows G'_5 .

We let $W = \{v_1, v_2, \dots, v_n\}$. Consider G_n which is G'_n with an arbitrary number k of isolated vertices added. Let **GF** be the natural greedy algorithm for ONLINE MAXIMAL FOREST, which accepts any vertex which does not create a cycle. If the adversary requests x and y before any vertex in W , **GF** cannot accept any vertex in W , so $\text{GF}(G'_n) = k + 2$. But there is another algorithm, **ALG**, which accepts more. The algorithm **ALG** accepts a vertex v if

- v has degree at most two and
- all neighbors of v have degree at least three (degree two before the current request).

We claim that **ALG** cannot accept both x and y . Assume x was requested before y and accepted. Now, y can only be accepted if x already has two other neighbors, v_i and v_j , when y is requested. However, this means that y will also have these two neighbors and **ALG** will reject it because its degree is at least three. The argument is symmetric if y is requested before x .

We now show that **ALG** accepts at least $k + n - 1$ vertices. The k isolated vertices will be accepted by **ALG** regardless of when they are requested.

Now assume x is requested before all vertices in W and before y . In this case, x is accepted. We have shown that y will be rejected when it is requested. At most two vertices from W can be rejected. When at least two vertices from W have already been requested and a new vertex v_i is requested, it holds that v_i has degree at most two and that all neighbors of v_i (x and possibly y) have degree at least three. Thus, v_i is accepted. In

total, at least $k + 1 + n - 2 = k + n - 1$ vertices are accepted. A symmetric situation holds if y is presented before x and all vertices in W .

We now consider the case where a vertex, $v_i \in W$ is requested before x and y . In this case, v_i is accepted. When x and y are requested, they will be rejected since they have a neighbor (v_i) whose degree it at most two. At most one vertex in W can be rejected, since after that, two vertices in W have already been requested (v_i was requested first). When another vertex v_j is requested, it holds that its possible neighbors (x and y) have degree at least three. In total, at least $k + n - 1$ vertices are accepted.

Hence, the greedy algorithm is not optimal for G_n with $n \geq 4$. $\square \quad \square$

We now consider another class of problems where the property does hold. This is a generalization of ONLINE INDEPENDENT SET. We consider the problem MAXIMUM ONLINE SET. In this problem, an instance consists of a base set E and a set of forbidden subsets $F \subseteq P(E)$. The forbidden subsets have the property that any superset of a forbidden subset is also forbidden. We let $x = \langle x_1, \dots, x_{|E|} \rangle$ denote the request sequence. There is a bijective function, f , mapping the x_i 's to E . For a set $S = \{x_i, x_j, x_h, \dots\}$, we let $f(S)$ denote $\{f(x_i), f(x_j), f(x_h), \dots\}$. This function f is not known to the algorithm. In request i , the algorithm receives request x_i . The request contains a list of all minimal subsets $A \subseteq \{x_1, \dots, x_{i-1}\}$ such that $f(A) \cup \{f(x_i)\} \in F$ (note that this list may be empty). The algorithm must reject or accept x_i . The produced solution is said to be feasible if it does not contain any subsets from F . The score of a feasible solution is the number of accepted elements. The score of an infeasible solution is $-\infty$. Note that if all minimal sets in F have size two, this is equivalent to ONLINE INDEPENDENT SET.

In MAXIMUM ONLINE SET, an isolated element is an element from E which is not in any sets of F . Note that such an element can be added to any solution. We let $s(E, F)$ denote a smallest $S \subseteq E$ such that adding any element to S results in a set which contains a forbidden subset.

The greedy algorithm, **GMOS**, is the algorithm which always accepts a request if the resulting solution is feasible.

For an algorithm, **ALG**, we let $\text{ALG}(E, F)$ be the smallest number such that there exist an ordering of E which causes **ALG** to accept at most $\text{ALG}(E, F)$ elements (using F as forbidden subsets). We let $MS^O(E, F)$ be the largest number such that there exists an algorithm with $\text{ALG}(E, F) = MS^O(E, F)$.

Theorem 5.4. *Let (E, F) be a MAXIMUM ONLINE SET instance, and let E' be E with the isolated elements removed. Let k denote the number of isolated*

elements. If $k + |s(E', F)| \geq MS^O(E', F)$, then

$$GMOS(E, F) = MS^O(E, F).$$

Proof. This proof is similar to that of Theorem 4.5. First note that **GMOS** accepts the k isolated elements and at least $|s(E', F)|$ elements from E' . For any algorithm the adversary can start by requesting elements, each with an empty list of forbidden sets it is already contained in. It continues until the algorithm has either accepted $|s(E', F)|$ elements or rejected k . The key argument is that the algorithm cannot distinguish between these initial elements.

If the algorithm accepts at least $|s(E', F)|$ elements, the adversary can decide that they were exactly those in $s(E', F)$, which **GMOS** also accepts. In this case, the algorithm cannot accept more than $k + |s(E', F)|$ elements in total.

If the algorithm rejects k elements, we need a result similar to that of Lemma 4.4. For **MAXIMUM ONLINE SET**, a pointless request is one, which reveals a forbidden set which contains only elements that have been accepted. Accepting a pointless request would result in an infeasible solution. The same argument as in the proof for Lemma 4.4 shows that an adversary loses no power by being conservative. Thus, when k elements have been rejected (and up to $|s(E', F)| - 1$ have been accepted), the adversary has a strategy for the remaining elements which ensures that the algorithm accepts at most $MS^O(E', F) \leq k + |s(E', F)|$ elements. \square \square

This problem is quite flexible. As we have mentioned, it can model independent set, but it could also model matroid intersection problems such as bipartite matching (though, even with more than two matroids). In this case, the forbidden sets, F , are the dependent sets in the union of the matroids.

6. Implications for Worst Case Performance Measures

Do the results from the previous section mean that **GIS** is a good algorithm for **ONLINE INDEPENDENT SET** if the input graph is known to be a Freckle Graph? The answer to this depends on how the performance of online algorithms is measured. In general, the answer is yes if a measure that only considers the worst case is used.

The most commonly used performance measure for online algorithms is *competitive analysis* [19]. For maximization problems, an algorithm, **ALG**, is

said to be c -competitive if there exists a constant, b , such that for any input sequence, I , $\text{OPT}(I) \leq c\text{ALG}(I) + b$ where $\text{OPT}(I)$ is the score of the optimal offline algorithm. For minimization problems, we require that $\text{ALG}(I) \leq c\text{OPT}(I) + b$. The competitive ratio of ALG is $\inf \{c : \text{ALG} \text{ is } c\text{-competitive}\}$. (Note that these ratios are always at least 1.) For *strict competitive analysis*, the definition is the same, except there is no additive constant.

Another measure is *on-line competitive analysis* [10], which was introduced for online graph coloring. The definition is the same as for competitive analysis except that $\text{OPT}(I)$ is replaced by $\text{OPTON}(I)$, which is the score of the best online algorithm that knows the requests in I but not their ordering. For graph problems, this means that the vertex-arrival model is used, as in this paper. The algorithm is allowed to know the final graph.

Corollary 6.1. *For ONLINE INDEPENDENT SET on Freckle Graphs, no algorithm has a smaller competitive ratio, strict competitive ratio, or on-line competitive ratio than GIS.*

Proof. Let ALG be a c -competitive algorithm for some c . Theorem 4.5 implies that GIS is also c -competitive. This argument also holds for the strict competitive ratio and the on-line competitive ratio. \square \square

Corollary 6.2. *For ONLINE VERTEX COVER on Freckle Graphs, no algorithm has a smaller competitive ratio, strict competitive ratio, or on-line competitive ratio than GVC.*

Corollary 6.3. *For ONLINE DOMINATING SET on the class of graphs with at least one isolated vertex, no algorithm has a smaller competitive ratio, strict competitive ratio, or on-line competitive ratio than GDS.*

Similar results hold for relative worst order analysis [4]. According to relative worst order analysis, for minimization problems in this graph model, one algorithm, A , is at least as good as another algorithm, B , on a graph class, if for all graphs G in the class, $A(G) \leq B(G)$. The inequality is reversed for maximization problems. It follows from the definitions that if an algorithm is optimal with respect to on-line competitive analysis, it is also optimal with respect to relative worst-order analysis. This was observed in [6]. Thus, the above results show that the three greedy algorithms in the corollaries above are also optimal on Freckle Graphs, under relative worst order analysis.

7. A Subclass of Freckle Graphs Where Greedy Is Not Optimal (Under Some Non-Worst Case Measures)

Although these greedy algorithms are optimal with respect to some worst case measures, this does not mean that these greedy algorithms are always the best choice for *all* Freckle Graphs. There is a subclass of Freckle Graphs where another algorithm is objectively better than **GIS**, and bijective analysis and average analysis [2] reflect this.

Theorem 7.1. *There exists an infinite class of Freckle Graphs $\tilde{G} = \{G_n \mid n \geq 2\}$ and an algorithm **Almost-GIS** such that for all $n \geq 2$ the following holds:*

$$\begin{aligned} \forall \phi \text{ \textbf{Almost-GIS}}(\phi(G_n)) &\geq \textbf{GIS}(\phi(G_n)) \\ \exists \phi \text{ \textbf{Almost-GIS}}(\phi(G_n)) &> \textbf{GIS}(\phi(G_n)) \end{aligned}$$

Proof. Consider the graph $G_n = (V_n, E_n)$, where

$$\begin{aligned} V_n &= \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z, u_1, u_2, \dots, u_n\} \\ E_n &= \{(x_i, y_i), (y_i, z), (z, u_i) \mid 1 \leq i \leq n\}. \end{aligned}$$

Figure 2 shows the graph G_4 .

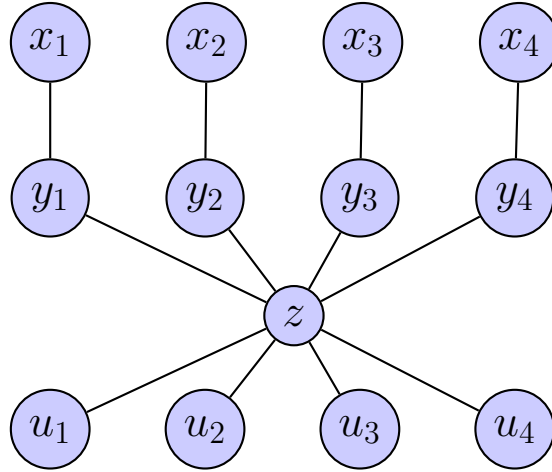


Figure 2: The graph G_4 .

We start by showing that G_n is a Freckle Graph. The smallest maximal independent set has size $n + 1$. We want to show that $I^O(G) = n + 1$, that is

no algorithm can get an independent set of size more than $n + 1$ in the worst case. We consider an arbitrary algorithm, **ALG**, and the situation where the adversary starts by presenting n isolated vertices. If **ALG** rejects all of these, the adversary can decide that it was u_1, \dots, u_n . In the remaining graph, it is not possible to accept more than $n + 1$ vertices. Otherwise, **ALG** accepts $i > 0$ of the $n + 1$ isolated vertices. The adversary can decide that one was z and that the remaining were x_1, \dots, x_{i-1} . Since **ALG** accepted z , it can never accept any of the vertices y_1, \dots, y_n or u_1, \dots, u_n . Thus, it can at most accept $n + 1$ vertices. This shows that G_n is a Freckle Graph.

The algorithm, **Almost-GIS**, is identical to **GIS**, except that it rejects a vertex if it already has two neighbors when it is presented. Consider any ordering of the vertices of G where **GIS** and **Almost-GIS** do not accept the same independent set. There must exist a first vertex, w , which is accepted by one of the algorithms and rejected by the other. By definition of the algorithms, it must be the case that w is rejected by **Almost-GIS** and accepted by **GIS**. It must hold that w has two neighbors, which have not been accepted by either algorithm. This can only happen if $w = z$ and the two neighbors are y_i and y_j where x_i and x_j have already been presented and accepted by both algorithms and no u_k have been presented yet. In this case, z is accepted by **GIS** and rejected by **Almost-GIS**. However, u_1, \dots, u_n are accepted by **Almost-GIS** and rejected by **GIS**. Since $n \geq 2$ and since both **GIS** and **Almost-GIS** accept exactly one of x_i and y_i , $1 \leq i \leq n$, we get that on every ordering, ϕ , where **GIS** and **Almost-GIS** accept a different independent set, $\text{Almost-GIS}(\phi(G)) > \text{GIS}(\phi(G))$. Such an ordering always exists (the ordering $x_1, \dots, x_n, y_1, \dots, y_n, z, u_1, \dots, u_n$ achieves this). \square \square

Competitive analysis, on-line competitive analysis, and relative worst order ratio do not identify **Almost-GIS** as a better algorithm than **GIS** on the class of graphs \tilde{G} defined in the proof of Theorem 7.1. There are, however, other measures which do this. Bijective analysis and average analysis [2] are such measures. Let I_n be the set of all input sequences of length $3n + 1$. Since we are considering the rather restricted graph class \tilde{G} , I_n denotes all orderings of the vertices in G_n (since these are the only inputs of length $3n + 1$). For an algorithm A to be considered better than another algorithm B for a maximization problem, it must hold for sufficiently large n that there exists a bijection $f : I_n \rightarrow I_n$ such that the following holds:

$$\begin{aligned}\forall I \in I_n \quad A(I) &\geq B(f(I)) \\ \exists I \in I_n \quad A(I) &> B(f(I))\end{aligned}$$

Theorem 7.2. *Almost-GIS is better than GIS on the class \tilde{G} according to bijective analysis.*

Proof. We let the bijection f be the identity and the result follows from Theorem 7.1. □ □

Average analysis is defined such that if one algorithm is better than another according to bijective analysis, it is also better according to average analysis. Thus, **Almost-GIS** is better than **GIS** on the class \tilde{G} according to average analysis.

Note that **Almost-GIS** is not an optimal algorithm for all Freckle Graphs. The class of graphs, $K_{n,n}$, for $n \geq 2$, consisting of complete bipartite graphs with n vertices in each side of the partition, is a class where **Almost-GIS** can behave very poorly. Note that on these graphs, **GIS** is optimal and always finds an independent set of size n , which is optimal, so these graphs are Freckle Graphs, even though they have no isolated vertices. If the first request to **Almost-GIS** is a vertex from one side of the partition and the next two are from the other side of the partition, **Almost-GIS** only accepts one vertex, not n .

8. Complexity of Determining the Online Independence Number, Vertex Cover Number, and Domination Number

Given a graph, G , it is easy to check if it has an isolated vertex and apply Theorem 4.9. However, Theorem 4.5 and Corollary 4.7 might not be as easy to apply, because it is not obvious how one can check if a graph is a Freckle Graph ($k + |s(G')| \geq I^O(G')$). In some cases, this is easy. For example, any graph where at least half the vertices are isolated is a Freckle Graph. We leave the hardness of recognizing Freckle Graphs as an open problem, but we show a hardness result for deciding if $I^O(G) \leq q$.

Theorem 8.1. *Given $q \in \mathbb{N}$ and a graph, G , deciding if $I^O(G) \leq q$ is NP-hard.*

Proof. Note that it is NP-complete to determine if the minimum maximal independent set of a graph, $G = (V, E)$, has size at most L , for an integer L [9]. To reduce from this problem, we create $\tilde{G} = (\tilde{V}, E)$ which is the same as G , but has $|V|$ extra isolated vertices, and a bound $\tilde{L} = L + |V|$. \tilde{G} is a Freckle Graph, since $|V| \geq I^O(G)$. By Corollary 4.6, $\text{GIS}(\tilde{G}) = I^O(\tilde{G})$. Since $\text{GIS}(\tilde{G}) = |s(G)| + |V|$, the original graph, G , has a minimum maximal independent set of size L , if and only if \tilde{G} has online independence number at most \tilde{L} . \square \square

The hardness of computing the online independence number implies the hardness of computing the online vertex cover number.

Corollary 8.2. *Given $q \in \mathbb{N}$ and a graph, G , deciding if $V^O(G) \geq q$ is NP-hard.*

Proof. This follows from Observation 2.2 and Theorem 8.1. \square \square

Theorem 8.3. *Given $q \in \mathbb{N}$ and a graph, G , deciding if $D^O(G) \geq q$ is NP-hard.*

Proof. We make a reduction from INDEPENDENT SET. In INDEPENDENT SET, a graph, G and an $L \in \mathbb{N}$ is given. It is a yes-instance if and only if there exists an independent set of size at least L . We reduce instances of INDEPENDENT SET, (G, L) , to instances of ONLINE DOMINATING SET, (\tilde{G}, \tilde{L}) , such that there exists an independent set in G of size at least L if and only if $D^O(\tilde{G}) \geq \tilde{L}$. The reduction is very simple. We let \tilde{G} be the graph which consists of G with one additional isolated vertex. We set $\tilde{L} = L + 1$. Assume first that any independent set in G has size at most $L - 1$. This means that any independent set in \tilde{G} has size at most L . Since GDS produces an independent set, it will accept at most $L < \tilde{L}$ vertices. Assume now that there is an independent set of size at least L in G . Then, there exists an independent set of size at least $L + 1$ in \tilde{G} . If these vertices are presented first, GDS will accept them. From Theorem 4.9, we get that no algorithm for ONLINE DOMINATING SET can do better (since \tilde{G} has an isolated vertex), which means that $D^O(\tilde{G}) \geq \tilde{L}$. \square \square

Theorem 8.4. *Given $q \in \mathbb{N}$ and a graph, G , the problem of deciding if $I^O(G) \leq q$ is in PSPACE.*

Proof. Let $q \in \mathbb{N}$ and a graph, $G = (V, E)$, be given. We sketch an algorithm that uses only polynomial space which decides if $I^O(G) \leq q$. We view the

problem as a game between the adversary and the algorithm where the algorithm wins if it gets an independent set of size at least $q + 1$. A move for the adversary is revealing a vertex along with edges to a subset of the previous vertices such that the resulting graph is an induced subgraph of G . These are possible to enumerate since induced subgraph can be solved in polynomial space. A move by the algorithm is accepting or rejecting that vertex. We make two observations: The game has only polynomial length (each game has length $2|V|$), and it is always possible in polynomial space to enumerate the possible moves from a game state. Thus, an algorithm can traverse the game tree using depth first search and recursively compute for each game state if the adversary or the algorithm has a winning strategy. \square \square

Similar proofs can be used to show that the problems of deciding if $V^O(G) \geq q$ and $D^O(G) \geq q$ are in PSPACE as well. It remains open whether these problems are NP-complete, PSPACE-complete, or neither. In [16] it was shown that determining the online chromatic number is PSPACE-complete if the graph is pre-colored and extended in [3] to hold even if the graph is not pre-colored.

9. Concluding Remarks

A strange difference between online and offline algorithms is observed: Adding isolated vertices to a graph can change an algorithm from not being optimal to being optimal (according to many measures). This holds for ONLINE INDEPENDENT SET, ONLINE VERTEX COVER, and ONLINE DOMINATING SET. It is also shown that adding isolated elements can make the natural greedy algorithm optimal for ONLINE MATCHING and MAXIMUM ONLINE SET (which includes ONLINE MATROID INTERSECTION as a special case), but not for all problems in the class AOC.

It is even more surprising that this difference occurs for vertex cover than for independent set, since in the offline case, adding isolated vertices to a graph can improve the approximation ratio in the case of the independent set problem. It is hard to see how adding isolated vertices to a graph could in any way help an offline algorithm for vertex cover.

We have shown that for Freckle Graphs, the greedy algorithm is optimal for ONLINE INDEPENDENT SET, but what about the converse? If a graph is not Freckle, is it the case that the greedy algorithm is not optimal? Let G be a graph, that is not a Freckle Graph. By definition, we have that

$I^O(G') > |s(G')| + k = \mathbf{GIS}(G)$. To show that the greedy algorithm is not optimal, we would have to show that $I^O(G) > \mathbf{GIS}(G)$. To show this, it would suffice to show that $I^O(G) \geq I^O(G')$. That is, the online independence number can never decrease when isolated vertices are added to a graph. We leave this as an open question.

Note that $\mathbf{GIS} = \mathbf{GDS}$. This means that for Freckle Graphs with at least one isolated vertex, \mathbf{GIS} is an algorithm which solves both online independent set (a maximization problem) and online dominating set (a minimization problem) online optimally. This is quite unusual, since the independent sets and dominating sets it will find in the worst case can be quite different for the same graph.

As mentioned earlier, the NP-hardness results presented here do not answer the question as to how hard it is to recognize Freckle Graphs. This is left as an open problem.

We have shown it to be NP-hard to decide if $I^O(G) \leq q$, $V^O(G) \geq q$, and $D^O(G) \geq q$, but there is nothing to suggest that these problems are contained in NP. They are in PSPACE, but it is left as an open problem if they are NP-complete, PSPACE-complete or somewhere in between.

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